Reanalysis of $e^{\pm}p$ Elastic Scattering Data in Terms of Proton Electromagnetic Formfactors

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Abstract

We have made the comparison of three parametrizations of the proton electromagnetic form factors in the space-like region using the largest data set on $e^{\pm}p$ elastic scattering ever used. All models have a correct analytical structure with a minimal set of well known singularities. For the first time an effective two-pion intermediate coulombic bound state (pionium) was included into GVMD for nucleon formfactors. For the proton charge radius an estimate derived from our best fits is as follows:

$$r_E^p[\text{fm}] = 0.897 \pm 0.002(\text{exp}) \pm 0.001(\text{norm}) \pm 0.003(\text{models}).$$

The traditional "model independent" fit by quadratic polynomial gives, in our estimation technology, a much cruder estimate

$$r^p_{E,polinomial}[\text{fm}] = 0.887 \pm 0.006(\text{exp}) \pm 0.096(\text{norm}).$$

Both estimates are closer to the value estimated from the recent hydrogen Lamb shift measurements.

Introduction

Since the discovery of the deviation of differential cross sections for ep elastic scattering [1] from that of calculated for point-like nucleons, numerous efforts were devoted to measurement of the "spatial extention" of nucleons and the corresponding electromagnetic form factors (FFs). There are more than 50 experimental reports with data on $e^{\pm}p$ elastic scattering, starting from a pioneering study of Hofstadter and co-workers in 1953-1956, up to recently reported new experimental data of Andivahis et al. in 1994 [10] (see Table 5.1).

Many attempts were made to model charge and magnetic moment distributions in the nucleons, numerous phenomenological parametrizations of (FFs) were proposed. The majority of proposed parametrizations is based on various modifications of generalized vector mesons dominance (GVMD) model guided by the requirements from general principles of quantum field theories and the modest amount is motivated by QCD features. The common features of the parameterizations are the requirements of providing reasonable values of the rms "charge" and "magnetic" radii of the nucleon ($t \to 0$ asymptotic), as well as reproducing QCD-like behaviour at large $|t|$.

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Unfortunately, it is still impossible to derive the FFs behaviour, even in the leading asymptotic terms, from the "first principles" and any extraction of derived physical parameters (such as radii and constants in an asymptotic behaviour at large $|t|$) would be "model dependent."

On the other hand, the precise knowledge of the nucleon FFs with as accurate as possible errors treatment is needed in a number of phenomenological applications and even in precise tests of well established theories. The analyses of charged hadrons elastic scattering in the Coulomb-nuclear interference regions and of the data on Lamb shifts in atomic hydrogen are the well known examples of such applications.

It is often argued that the extraction of FFs from the experimental data is an ill posed problem. The phenomenological basis of experimental FFs extraction is the well known Rosenbluth formula for $d\sigma/dt$ in the one photon exchange approximation [11]. Different experimental groups have extracted FFs from their own data on elastic scattering with loose information on the sources and on the nature of different errors estimated for differential cross sections. As a result we have no reference numbers to make all the data on FFs extracted by different groups mutually consistent.

As the common practice of usage the experimental information on FFs in applications is via the "currently best" parametrizations we propose in this paper a schema to select the best FFs parametrizations by fitting to the world $e^{\pm}p$ elastic scattering data without using the derived data on FFs. It makes possible to have a relatively complete control of the error matrices of the technical and physical parameters and to estimate the errors induced by switching of the competitive parametrizations and errors induced by our method used to make world database self consistent in relative normalizations of different experiments.

In this paper we analyze only $e^{\pm}p$ elastic scattering data. The article is organized as follows.

Section 1. Here we reproduce a compilation of traditional formulas used in data analyses for extraction of the FFs from the data on $d\sigma/dt$.

Section 2. presents the description of our preparatory handling of the experimental data published so far and stored in the ReacData [27] database of PPDS system. Our procedure of the cross assessments of data and models is also outlined in this section and in the Appendices.

Section 3. is a compilation of the parameterizations used in cross assessments.

Section 4. Here we describe the preparatory fit results for each model selected for cross assessments.

Section 5. gives details of our models driven procedures to make data from different experiments consistent with each other.

Section 6. Here we describe the procedure of the model-dependent filtration and present the final results of the fits to the filtered data.

Section 7. The summary of the results obtained and the outline of the future iterations of assessments are given in this section.

1 General description of the $e^{\pm}p$ elastic scattering

The differential cross section of the elastic $e^{\pm}p$ scattering in the laboratory reference frame is parameterized by the well known Rosenbluth formula [11]

$$\frac{d\sigma}{d\Omega} = \left\{\frac{d\sigma}{d\Omega}\right\}_{ns} \cdot \left[\frac{\tau G_m^2 + G_E^2}{1 + \tau} + 2\tau G_M^2 tg^2(\theta/2)\right], \tag{1}$$

where

$$\tau = \frac{q^2}{4m_p^2}, \quad \left\{\frac{d\sigma}{d\Omega}\right\}_{ns} = \left(\frac{\alpha}{2E}\right)^2 \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \cdot \left\{1 + \frac{2E}{m_p}\sin^2(\theta/2)\right\}^{-1},$$

E is the incoming electron energy, θ is the scattering angle of the electron in the lab frame, and m_p is the proton mass.

The Rosenbluth formula is based on the one photon exchange approximation QED diagram with modified proton-photon-proton vertex for the extended proton

$$\Gamma_\mu(q^2) = \gamma_\mu F_1(q^2) + i\sigma_{\mu\nu}\frac{q_\nu}{2m_p}F_2(q^2),$$

with Dirac (F_1) and Pauli (F_2) FFs introduced. These FFs are connected with so called "Sachs" FFs [4] as follows:

$$G_E^{p,n}(q^2) = F_1^{p,n}(q^2) - \frac{q^2}{4m_{p,n}^2}F_2^{p,n}(q^2), \quad G_M^{p,n}(q^2) = F_1^{p,n}(q^2) + F_2^{p,n}(q^2).$$

In the non-relativistic limit they describe an electric charge and magnetic moment distributions within the proton with the normalizations as follows:

$$\begin{aligned} G_E^p(0) &= 1, \qquad & G_M^p(0) &= \mu_p, \\ G_E^n(0) &= 0, \qquad & G_M^n(0) &= \mu_n, \end{aligned}$$

where $\mu_{p,n}$ are the (p,n) magnetic moments and

$$\langle r_E^p \rangle^2 = -6 \cdot \left. \frac{dG_E^p(q^2)}{dq^2} \right|_{q^2=0}$$

is the mean square radius of the proton charge distribution in units $[GeV^{-2}]$. In terms of invariant variables s and t the formula for differential cross section takes the form

$$\frac{d\sigma}{d(-t)} = \frac{4\pi\alpha^2}{t^2}\left(1+\frac{t}{\tilde{s}}+\frac{tm_p^2}{\tilde{s}^2}\right)\left[\frac{G_E^2+\tau G_M^2}{1+\tau}+\frac{G_M^2}{2}\frac{t^2}{\tilde{s}(\tilde{s}+t)+tm_p^2}\right], \tag{2}$$

where $\tilde{s} = s - m_p^2, \quad t = -q^2 = -4E^2\sin^2(\theta/2)\cdot[1+\frac{2E}{m_p}\sin^2(\theta/2)]^{-1}$. Note, that t varies in the range $-\tilde{s}^2/s < t < 0$. When $s \to \infty$, then one has

$$\frac{d\sigma}{d(-t)} \to \frac{4\pi\alpha^2}{t^2}\cdot\left[\frac{G_E^2+\tau G_M^2}{1+\tau}\right], \tag{3}$$

for any fixed value of t. It can be shown that the righthand side of (3) is an upper envelope curve for all $\frac{d\sigma}{d(-t)}(s,t)$.

As is seen from Fig. 1, this is confirmed experimentally in the wide ranges of both variables s and t. This is a direct evidence that one photon approximation works well at the available level of accuracy of the world experimental data.

To model FFs, we follow a conventional way with introduction of the normalized to unity at $t = 0$ isospin FFs, namely, $F_{1,2}^{s,v}$. They are connected with the $F_{1,2}^{p.n}$ and $G_{E,M}^{p,n}$ through the relations:

$$F_1^p = \frac{1}{2}\left(F_1^s + F_1^v\right), \quad F_2^p = \frac{\tilde{\mu}_p+\tilde{\mu}_n}{2}F_2^s + \frac{\tilde{\mu}_p-\tilde{\mu}_n}{2}F_2^v,$$

$$F_1^n = \frac{1}{2}\left(F_1^s - F_1^v\right), \quad F_2^n = \frac{\tilde{\mu}_p+\tilde{\mu}_n}{2}F_2^s - \frac{\tilde{\mu}_p-\tilde{\mu}_n,}{2}F_2^v,$$

where $\tilde{\mu}_{p,n}$ are the anomalous magtnetic moments of proton and neutron.

2 Data sample

We use the total sample of data on differential cross-sections for the electron-proton elastic scattering extracted from the PPDS ReacData data base [27] (see also Reaction Data [28]). The complete list of references for the used experimental data is presented in Table 5.1.

The data are divided by the sets corresponding to different experiments or experimental methodics. For example, if the different methodics were used in the same work to obtain the data, we treat different data samples as independent because of possible difference in systematic normalization errors. The data of sets 34 and 35 are taken from the graphs. In all the cases, where it was possible, we separated the systematic normalization errors from the total ones in order to renormalize the different data sets in a such a way that makes the whole compilation self-consistent. In a few cases we failed to separate the normalization errors from the total errors. We classify such data sets as "incomplete data" and will use them only at the stage of the first crude adjustments of the models. The incomplete data sets are sets 2,3,18,19,23,24,41 of Table 5.1. They have only the total errors assigned without detailed information about systematic errors.

We have also excluded 5 points of set 1 (BUMILLER 61, PR 124, 1623, see Table 2.1.) that have inconsistent information about the errors, and few points that turned out to be in sharp contradictions with the nearest experimental points and are out of all parametrizations by more than 5-6 standard deviations.

Table 2.1. Excluded points

	Experiment	$q^2[GeV^2]$	$E[GeV]$	$\theta[deg]$
1	BUMILLER 61	0.76	0.700	145
		0.45	0.800	60
		0.33	0.850	45
		0.52	0.875	60
		0.70	0.900	75
5	YOUNT 62	0.202	0.307	130
6	+YOUNT 62	0.202	0.307	130
25	MISTRETTA 69	0.422	4.529	
26	BREIDENBACH 70		7.	6
37	BORKOWSKY 74	$0.76[fm^{-2}]$	0.1498	
		$0.56[fm^{-2}]$	0.1801	

The resulting data sample that we have included in the first crude analysis (see section 5) consists of 847 experimental points covering the q^2 range from 0.005 to 31.280 GeV^2 and 1961-1994 time period (see Fig. 1).

3 Models

To construct a simple but flexible enough estimator for the nucleon electromagnetic FFs, we explore the majority of previous old and recent ideas invented in the papers devoted to FFs modelling and data analyses [1],[2] – [24]: the simplest analytic structure of Pauli and Dirac FFs with the only established singularities that have a clear physical interpretation and with an asymptotic behaviour at a large momentum transfer as predicted by perturbative QCD [25]. Isoscalar and isovector FFs[1] are modelled as follows:

$$F^{s,v}_{1,2}(t) \quad = \quad F^{s,v}_{GVMD(1,2)}(t) \times A^{s,v}_{1,2}(t), \tag{1}$$

where $F^{s,v}_{GVMD(1,2)}(t)$ are the generalized vector dominance model factors, and $A^{s,v}_{1,2}$ are the terms which control the asymptotic behavior of FFs at $(-t) \to \infty$.

All the functions used are the real analytic functions with real magnitudes at $t < 0$, with cuts dictated by the first few particles thresholds at $t > 0$. $F^{s,v}_{GVMD(1,2)}(t)$ have resonant poles only on secondary sheets of the Riemann surface of the FF and constructed as a sum of individual contributions corresponding to resonant poles.

$\boxed{\boldsymbol{f_{pionium}(t)}}$ is the term motivated by possible poles of pionium spectrum (corresponding to 1^{--} electromagnetic bound states of the $\pi^+\pi^-$ system). There is no possibility to get a parametrization of the whole pionium stectrum contribution to the FFs. Therefore, we introduce only the nearest to the point $t = 0$ pole contribution to the FFs

$$f_{pionium}(t) \quad = \quad \frac{(\sqrt{1 - t/4m_\pi^2}) \cdot (1 + \sqrt{1 - (2m_\pi - \epsilon_\pi)^2/4m_\pi^2})}{\sqrt{1 - t/4m_\pi^2} + \sqrt{1 - (2m_\pi - \epsilon_\pi)^2/4m_\pi^2}}. \tag{2}$$

It is the real analytic function in the whole t-plane with a righthand cut $[4m_\pi^2, \infty[$ with a pole on the second Rieman sheet. It corresponds to the lowest 1^{--} pionium "bound" state with the

[1]Mathematica and Fortran codes of all the models can be obtained by request from: polishchuk@mx.ihep.su.

binding energy $-\epsilon_\pi$, which is unstable under annihilation. We neglect the pionium annihilation width in form (2). By construction it has the following properties:

$$f_{pionium}(0) = 1, \quad f_{pionium}(t) \to 1 \quad \text{at} \quad |t| \to \infty, \quad f_{pionium}(t)|_{\epsilon_\pi = 0} \equiv 1.$$

$\boxed{\boldsymbol{f^\rho(t)}}$ is the model of ρ meson pole contribution to $F^v_{GVMD(1,2)}(t)$ in the whole t-plane. It is constructed from the real analytic function $\Phi^\rho(t)$, proposed by Wataghin[15]. It has the cut $[4m_\pi^2, \infty[$ and two poles in the unphysical Riemann sheets in the complex conjugated points

$$\Phi^\rho(t) = \frac{m_\rho \Gamma_\rho}{m_\rho^2 - t + m_\rho \Gamma_\rho \cdot \sqrt{\frac{4m_\pi^2 - t}{m_\rho^2 - 4m_\pi^2}}}, \tag{3}$$

where ρ pole positions in two complex conjugated points are both on the unphysical sheet as required by real analyticity. The pole positions are determined by requirement to reproduce the Breit-Wigner shape in the vicinity of ρ resonance ($t = m_\rho^2$) at the upper edge of the cut in the small width approximation. The nonzero width ρ meson contributions to the $F^v_{GVMD(1,2)}(t)$ is given by

$$f^\rho(t) = \frac{\Phi^\rho(t)}{\Phi^{\rho)}(0)} \tag{4}$$

with the following properties

$$f^\rho(0) = 1, \qquad f^\rho(t) \sim \frac{m_\rho \Gamma_\rho}{t} \quad \text{at} \quad |t| \to \infty.$$

$\boxed{\boldsymbol{f^\omega_{1,2}(t)}}$ are the models of ω meson contributions to FFs in the whole t-plane with the righthand cuts. They are the real analytic finctions with cuts $[4m_\pi^2, \infty[$ and $[9m_\pi^2, \infty[$ with poles on the unphysical Riemann sheets in the complex conjugated points, and are constructed as follows[2] :

$$\begin{aligned} \Phi^\omega_{1,2}(t) &= \cos\Theta^\omega_{1,2} \cdot \frac{m_\omega \Gamma^{(3)}_\omega}{m_\omega^2 - t + m_\omega \Gamma_\omega / \pi \cdot \ln \frac{9m_\pi^2 - t}{m_\omega^2 - 9m_\pi^2}} \\ &+ \sin\Theta^\omega_{1,2} \cdot \frac{m_\omega \Gamma^{(2)}_\omega}{m_\omega^2 - t + m_\omega \Gamma_\omega \cdot \sqrt{\frac{4m_\pi^2 - t}{m_\omega^2 - 4m_\pi^2}}}, \end{aligned} \tag{5}$$

where ω resonance pole positions are in the complex conjugated points on the unphysical sheets corresponding to two and three body branch cuts. The pole positions are determined by the requirement to reproduce the Breit-Wigner shape in the vicinity of resonance on the upper edge of the cut in the small width approximation. The nonzero width omega meson contributions to the FFs are as follows:

$$f^\omega_{1,2}(t) = \frac{\Phi^\omega_{1,2}(t)}{\Phi^\omega_{1,2}(0)}, \tag{6}$$

with properties

$$f^\omega_{1,2}(0) = 1, \qquad f^\omega_{1,2}(t) \sim \frac{const}{t} \quad \text{at} \quad |t| \to \infty.$$

[2]Here we try to test whether differrent branch channels can be identified from the fit. So, our form for ω and ϕ contributions differs from that used in [15],[22],[24].

$\boxed{f^{\phi}_{1,2}(t)}$ are the models of ϕ meson contributions to FFs in the whole t-plane with the righthand cuts. They are the real analytic finctions with cuts $[4m_K^2,\infty[$ and $[9m_\pi^2,\infty[$ with poles on the unphysical Riemann sheets in the complex conjugated points. They are constructed in a similar way

$$\begin{aligned}\Phi^{\phi}_{1,2}(t) &= \cos\Theta^{\phi}_{1,2}\cdot\frac{m_\phi\Gamma^{(3)}_\phi}{m_\phi^2-t+m_\phi\Gamma_\phi/\pi\cdot\ln\frac{9m_\pi^2-t}{m_\phi^2-9m_\pi^2}}\\ &+ \sin\Theta^{\phi}_{1,2}\cdot\frac{m_\phi\Gamma^{(2)}_\phi}{m_\phi^2-t+m_\phi\Gamma_\phi\cdot\sqrt{\frac{4m_K^2-t}{m_\phi^2-4m_K^2}}}.\end{aligned}\tag{7}$$

The nonzero width ϕ meson contributions to the FFs have the form

$$f^{\phi}_{1,2}(t) = \frac{\Phi^{\phi}_{1,2}(t)}{\Phi^{\phi}_{1,2}(0)},\tag{8}$$

with properties

$$f^{\phi}_{1,2}(0)=1,\qquad f^{\phi}_{1,2}(t)\sim\frac{const}{t}\quad\text{at}\quad |t|\to\infty.$$

Other ingredients of our hybrid model are multiplicative factors $A^{s,v}_{1,2}$ which control the FFs behaviour at large $|t|$. We use the two forms for it.

$\boxed{M^{s,v}_{1,2}(t)}$ are motivated by the parametrization of Mack [13]

$$M^{s,v}_i(t) = \left(\frac{1}{1-t/t^p_{cut}}\right)^{x^{s,v}_1+i-1}\left(\frac{1}{1-t/t^{s,v}_{cut}}\right)^{x^{s,v}_2\ln\frac{(t^{s,v}_{cut}-t)}{\lambda^2}},\quad i=1,2,\tag{9}$$

where $t^p_{cut}=4m_p^2$ is the proton-antiproton threshold, $t^s_{cut}=9m_\pi^2$, $t^v_{cut}=4m_\pi^2$, and λ is the scale parameter which is expected to be of the order of the Λ_{QCD}.

$M^{s,v}_{1,2}(t)$ are the real analytic functions with singularities only on the righthand cuts $[t^{s,v}_{cut},\infty[$ and $[4m_p^2,\infty[$, normalized by $M^{s,v}_{1,2}(0)=1$. At large $-t$ they decrease faster than predicted by QCD, but they give quite good fits at available t as it will be shown further. This model is included in the current analysis in order to test whether the QCD asymptotic is really prominent at the available highest q^2.

$\boxed{GK^{s,v}_{1,2}(t)}$ are motivated by parametrization of Gari and Krümpelman [18] with taking into account the QCD asymptotic logarithms by analogy with the parametrizations used in [20],[23]

$$GK^{s,v}_i(t) = \frac{(1-t/t^p_{cut})^{\nu+1-i}}{\left(1-\eta^{s,v}_i+\eta^{s,v}_i(1-t/t^{s,v}_{cut})\ln\left(\frac{t^{s,v}_{cut}-t}{\lambda^2}\right)/\ln\frac{t^{s,v}_{cut}}{\lambda^2}\right)^{2+\nu}},\quad i=1,2,\tag{10}$$

$GK^{s,v}_i(t)$ are the real analytic finctions with singularities only on the righthand cuts $[t^{s,v}_{cut},\infty[$ and $[4m_p^2,\infty[$, normalized by $GK^{s,v}_{1,2}(0)=1$ at $t=0$. At large $-t$, they reproduce the FFs behaviour at large $-t$ as predicted by pQCD. The exponent ν is given [25] by

$$\nu=\frac{4}{3\beta},\quad \beta=11-\frac{2}{3}N_f,\quad \nu=0.174\quad\text{for}\quad N_f=5.$$

Now we are ready to assemble parametrizations for isotopis form factors.

Modified Mack-67 parametrization (Mack-98)

$$
\begin{aligned}
F^s_{1,2} &= \big((1-C^{\omega}_{1,2})\cdot f_{pionium}(t) + C^{\omega}_{1,2}\cdot f^{\omega}_{1,2}(t)\big)\times M^s_{1,2}, \\
F^v_{1,2} &= \Big((1-C^{\rho}_{1,2})\cdot f_{pionium}(t) + C^{\rho}_{1,2}\cdot f^{\rho}(t)\Big)\times M^v_{1,2}
\end{aligned}
\tag{11}
$$

with 10 adjustable parameters: $C^{\rho,\omega}_{1,2}, \quad \Theta^{\omega}_{1,2}, \quad x^{s,v}_{1,2}$.

Modified Gari-Krümpelman parametrization (GK-98-2V)

$$
\begin{aligned}
F^s_{1,2} &= \big((1-C^{\omega}_{1,2})\cdot f_{pionium}(t) + C^{\omega}_{1,2}\cdot f^{\omega}_{1,2}(t)\big)\times GK^s_{1,2}(t), \\
F^v_{1,2} &= \Big((1-C^{\rho}_{1,2})\cdot f_{pionium}(t) + C^{\rho}_{1,2}\cdot f^{\rho}(t)\Big)\times GK^v_{1,2}
\end{aligned}
\tag{12}
$$

with 10 adjustable parameters: $C^{\rho,\omega}_{1,2}, \quad \Theta^{\omega}_{1,2}, \quad \eta^{s,v}_{1,2}$.

Modified Gari-Krümpelman parametrization (GK-98-3V))

$$
\begin{aligned}
F^s_{1,2} &= \Big((1-C^{\omega}_{1,2}-C^{\phi}_{1,2})\cdot f_{pionium}(t) + C^{\omega}_{1,2}\cdot f^{\omega}_{1,2}(t) + C^{\phi}_{1,2}\cdot f^{\phi}_{1,2}(t)\Big)\times GK^s_{1,2}(t), \\
F^v_{1,2} &= \Big((1-C^{\rho}_{1,2})\cdot f_{pionium}(t) + C^{\rho}_{1,2}\cdot f^{\rho}(t)\Big)\times GK^v_{1,2}
\end{aligned}
\tag{13}
$$

with 14 adjustable parameters: $C^{\rho,\omega,\phi}_{1,2}, \quad \Theta^{\omega}_{1,2}, \quad \Theta^{\phi}_{1,2}, \quad \eta^{s,v}_{1,2}$.

4 Preliminary fit results

To select "the best models" for the FFs, we start with the crude fits of each above parametrization to the overall data sample (847 data points, see Table 5.1.). We used a mean least squares method with weights defined by the "total error" – the quadratically combined all kinds of experimental errors, ignoring correlations. The results of our best fits of analytic FFs are given in Table 4.1. The best fit estimations for the parameters and two kinds of their error estimates are presented: 1) the Minuit errors, 2) the errors obtained by propagation of the combined errors of the experimental data points to the parameter errors using equations to find the position of minimum in the parameter space. Since we will use the error propagation technique to get separate estimates of the parameter errors of different kinds due to statistical, systematic, and normalization errors in the input data, we must be sure that we are in a local minimum and parameters estimates are sufficiently close to the minimum position. These two errors must be close to each other if the usual Minuit assumptions about the minimum are valid and if the distance to the true minimum is sufficiently small (see Appendix 2).

Table 4.1. All data, weights made from all errors combined in quadratures.

Model name & Fit quality	Free & derived parameters	Estimated parameter values and their errors
Mack-98	x_1^s	2.702± 0.024(0.024)
	x_1^v	0.894 ± 0.032(0.031)
	x_2^s	0.000 (fixed)
	x_2^v	0.05232 ± 0.00047(0.00047)
χ^2/dof	C_1^ρ	0.9681 ± 0.0069(0.0068)
986.4/836	C_2^ρ	0.9127 ± 0.0049(0.0048)
1.179	C_1^ω	0.5701 ± 0.0071(0.0071)
	C_2^ω	−0.75 ± 0.17(0.17)
	θ_1^ω[deg]	91.541± 0.069(0.079)
	θ_2^ω[deg]	91.41413± 0.00013(0.00014)
	λ[GeV]	0.1024 ± 0.0039(0.0039)
	r_E^p[fm]	**0.860 ± 0.003(0.003)**
GK-98-2V	η_1^v	0.06699 ± 0.00142(0.00141)
	η_2^v	0.0364 ± 0.0011(0.0011)
	η_1^s	0.1191 ± 0.0027(0.0027)
	η_2^s	0.0471 ± 0.0029(0.0029)
χ^2/dof	C_1^ρ	2.380 ± 0.041(0.041)
973.6/836	C_2^ρ	1.0843 ± 0.0093(0.0093)
1.165	C_1^ω	−1.952 ± 0.038(0.038)
	C_2^ω	8.01 ± 0.18(0.18)
	θ_1^ω[deg]	91.3777 ± 0.0021(0.0021)
	θ_2^ω[deg]	91.40405 ± 0.00053(0.00053)
	λ[GeV]	0.0133 ± 0.0050(0.0050)
	r_E^p[fm]	**0.865 ± 0.002(0.002)**
GK-98-3V	η_1^v	0.06806 ± 0.00099(0.00099)
	η_2^v	0.03002 ± 0.00046(0.00047)
	η_1^s	0.10627 ± 0.00048(0.00048)
	η_2^s	0.04876 ±0.00057(0.00057)
χ^2/dof	C_1^ρ	2.312 ± 0.026(0.023)
972.4/833	C_2^ρ	1.2383 ± 0.0070(0.0059)
1.167	C_1^ω	−2.583 ± 0.023(0.021)
	C_2^ω	8.99 ± 0.13(0.07)
	C_1^ϕ	1.290 ± 0.022(0.022)
	C_2^ϕ	1.91 ± 0.18(0.19)
	θ_1^ω[deg]	91.3779 ± 0.0029(0.0029)
	θ_2^ω[deg]	91.40402 ± 0.00097(0.00097)
	θ_1^ϕ[deg]	157.98 ± 0.99(0.99)
	θ_2^ϕ[deg]	157.95 ± 8.15(8.11)
	λ[GeV]	0.0136 ± 0.0046(0.0046)
	r_E^p[fm]	**0.861 ± 0.002(0.002)**
Polynomial	a_1	−0.241 ± 0.005(0.005)
χ^2/dof	a_2	0.023 ± 0.009(0.009)
0.928	r_E^p[fm]	**0.849 ± 0.008(0.008)**

In addition to the parametrizations under consideration we will also use the traditional second order polynomial

$$1 + a_1(\frac{q^2}{4m_\pi^2}) + a_2(\frac{q^2}{4m_\pi^2})^2$$

in the q^2 range from 0 to 0.06 GeV^2. Here $4m_\pi^2 = 0.078\ GeV^2$ is the bound of convergence circle for the presentation of FF by power series; a_1 and a_2 are the free parameters. This parametrization is often used to estimate the proton electric radius. It is sensitive to the behaviour of FFs near a zero value of q^2 and large deviation of "polynomial" value from predictions of other models means that smooth parametrizations do not reproduce a possible fine structure near $q^2 = 0$ and probably may give wrong results for proton radius. On the other hand, such a "fine structure" might be caused by inconsistency in normalizations of different data sets in this region. Such an inconsistency of the existing data in normalizations cannot be removed without using smooth parametrizations. Or we will have to wait until a new single experiment will precisely and densely enough measure the range of small q^2, sufficient to get reliable polynomial fits.

It should be noted that we obtain the best quality fits of the whole data sample of the $d\sigma/dt$ by each of our three analytic models ever obtained by other analyses with other models. We quote in the tables the parameters errors estimated by two independent methods as it was described above.

But all these fits are still not good enough to draw conclusions about the reliability of the physical parameters estimates (in spite of that we have obtained the estimates of the proton electric radius in a good agreement with the latest published estimates [23]).

It should be stressed also that the model with three vector mesons has no advantages in describing the data in the space-like region of these naive fits. Pionium, ρ, and ω contributions are sufficient to get a relatively good data description.

Let us try to get more reliable fits by excluding correlated normalization errors from the weights, because this may cause (as we will show further) a significant biases in parameter estimates. In the next section we use the correlated normalization uncertainties to make data sets relatively consistent with each other on the basis of the selected models.

5 Data normalizing

For further analyses we have excluded the so called "incomplete data sets": 2, 3, 18, 19, 23, 24, 41 from the total sample of the data (see Table 5.1). They do not contain any information about systematics, and cannot be used in the procedure of renormalizations. To uncover a possible effect of this formal filtration we repeate the previous fit on the reduced sample of experimental data with the results presented in Table 5.2. As is seen we obtain quite good fit qualities for all the models and parameters shifted only slightly. As for the proton radii, they are not practically shifted.

Having obtained a good enough fits by all the parameterizations, we remain on the firm ground to discuss the values of the proton radius. As is seen from Table 5.2, the model dependent radii are close to each other with overlapping error bands, but the radius obtained from a polynomial fit is smaller and its interval estimator,

$$r_E^p \in [0.841, 0.857][\text{fm}],$$

does not overlap with a joined interval estimator of model radii, namely

$$r_E^p \in [0.858, 0.867][\text{fm}].$$

Table 5.1. Statistics of data sets, renormalization factors, and references.

Set No	Set number of points	After filtration	Normalization factors	Points in 90% CL sample	REACDATA ShortCode	Reference
1	53	53	0.985 ± 0.002	45	BUMILLER 61	PR 124, 1623
2	9	0	–	0	OLSON 61	PRL 6, 286
3	20	0	–	0	LITTAUER 61	PRL 7, 141
4	2	2	0.999 ± 0.003	0	LEHMANN 62	PR 126, 1183
5	3	3	1.009 ± 0.004	3	YOUNT 62	PR 128, 1842
6	3	3	1.000 ± 0.000	3	+ YOUNT 62	PR 128, 1842
7	8	8	1.017 ± 0.004	7	DRICKEY 62	PRL 9, 521
8	6	6	1.006 ± 0.001	4	DUDELZAK 63	NC 28, 18
9	21	21	1.011 ± 0.011	17	BERKELMAN 63	PR 130, 2061
10	7	7	0.988 ± 0.013	5	DUNNING 63	PRL 10, 500
11	6	6	1.030 ± 0.012	5	CHEN 65	PR 141, 1267
12	12	12	1.052 ± 0.009	8	+ CHEN 65	PR 141, 1267
13	10	10	1.000 ± 0.000	8	FREREJACQUE 65	PR 141, 1308
14	3	3	1.008 ± 0.006	3	GROSSETETE 65	PR 141, 1435
15	93	93	1.000 ± 0.000	88	JANSSENS 66	PR 142, 922
16	3	3	1.022 ± 0.005	0	BENAKSAS 66	PR 148, 1327
17	11	11	1.000 ± 0.000	10	BARTEL 66	PRL 17, 608
18	10	0	–	0	ALBRECHT 66	PRL 17, 1192
19	7	0	–	0	+ ALBRECHT 66	PRL 17, 1192
20	24	24	0.986 ± 0.004	20	BEHREND 67	NC A48, 140
21	4	4	0.912 ± 0.013	1	+ BEHREND 67	NC A48, 140
22	1	1	1.009 ± 0.015	1	BARTEL 67	PL 25B, 236
23	4	0	–	0	BARTEL 67B	PL 25B, 242
24	3	0	–	0	ALBRECHT 67	PRL 18, 1014
25	4	4	0.997 ± 0.003	3	MISTRETTA 69	PR 184, 1487
26	8	8	1.003 ± 0.009	8	BREIDENBACH 70	MIT-2098-635
27	22	22	1.004 ± 0.003	18	LITT 70	PL 31B, 40
28	15	15	1.002 ± 0.003	11	GOITEIN 70	PR D1, 2449
29	54	54	0.996 ± 0.002	52	BERGER 71	PL 35B, 87
30	9	9	1.005 ± 0.003	9	PRICE 71B	PR D4, 45
31	4	4	0.985 ± 0.012	4	GANICHOT 72	NP A178, 545
32	5	5	1.003 ± 0.004	5	BARTEL 73	NP B58, 429
33	16	16	0.997 ± 0.003	15	+ BARTEL 73	NP B58, 429
34	37	37	0.992 ± 0.001	35	BOTTERILL 73C	PL 46B, 125
35	30	30	0.988 ± 0.001	22	+ BOTTERILL 73C	PL 46B, 125
36	11	11	0.997 ± 0.006	10	KIRK 73	PR D8, 63
37	44	44	0.994 ± 0.001	37	BORKOWSKY 74	NP A222, 269
38	11	11	1.013 ± 0.002	11	MURPHY 74	PR C9, 2125
39	77	77	0.996 ± 0.001	69	BORKOWSKY 75	NP B93, 461
40	7	7	0.993 ± 0.004	7	STEIN 75	PR D12, 1884
41	3	0	–	0	MARTIN 76C	PRL 38, 1320
42	8	8	1.022 ± 0.008	8	MESTAYER 78	SLAC-214
43	49	49	1.002 ± 0.000	40	AKIMOV 78B	YF 29, 922
44	5	5	1.002 ± 0.001	5	SIMON 80	NP A333, 381
45	22	22	0.998 ± 0.002	21	WALKER 89	PL 224B, 353
46	11	11	0.995 ± 0.005	11	BOSTED 89	PR C42, 38
47	5	5	0.969 ± 0.011	5	ROCK 91	PR D46, 24
48	13	13	0.994 ± 0.004	12	SILL 93	PR D48, 29
49	22	22	0.998 ± 0.002	22	WALKER 94	PR D49, 5671
50	32	32	1.000 ± 0.002	28	ANDIVAHIS 94	PR D50, 5491
All	**847**	**791**		**696**		

Table 5.2. Fits to sets of complete data only, weights made from all errors combined in quadratures.

Model name & Fit quality	Free & derived parameters	Estimated parameter values and their errors
Mack-98	x_1^s	2.587 ± 0.091(0.079)
	x_1^v	0.978 ± 0.072(0.061)
	x_2^s	0.0035 ± 0.0029(0.0027)
	x_2^v	0.0505 ± 0.0017(0.0015)
	C_1^ρ	0.9665 ± 0.0072(0.0071)
χ^2/dof	C_2^ρ	0.9063 ± 0.0051(0.0050)
757.4/780	C_1^ω	0.5767 ± 0.0073(0.0072)
0.971	C_2^ω	−0.97 ± 0.17(0.17)
	θ_1^ω[deg]	91.523 ± 0.045(0.044)
	θ_2^ω[deg]	91.414162 ± 0.000085(0.000084)
	λ[GeV]	0.1010 ± 0.0039(0.0039)
	r_E^p[fm]	**0.861 ± 0.003(0.003)**
GK-98-2V	η_1^v	0.0624 ± 0.0015(0.0015)
	η_2^v	0.02762 ± 0.00090(0.00090)
	η_1^s	0.1094 ± 0.0027(0.0027)
	η_2^s	0.0427 ±0.0023(0.0023)
χ^2/dof	C_1^ρ	2.317 ± 0.038(0.039)
750.4/780	C_2^ρ	1.1894 ± 0.0090(0.0090)
0.962	C_1^ω	−1.723 ± 0.035(0.035)
	C_2^ω	9.01 ± 0.19(0.19)
	θ_1^ω[deg]	91.3779 ± 0.0023(0.0023)
	θ_2^ω[deg]	91.40402 ± 0.00047(0.00047)
	λ[GeV]	0.0194 ± 0.0064(0.0064)
	r_E^p[fm]	**0.865 ± 0.002(0.002)**
GK-98-3V	η_1^v	0.0505 ± 0.0014(0.0014)
	η_2^v	0.02281 ± 0.00072(0.00073)
	η_1^s	0.0989 ± 0.0020(0.0020)
	η_2^s	0.0437 ±0.0017(0.0017)
χ^2/dof	C_1^ρ	3.3392 ± 0.0092(0.0092)
747.9/776	C_2^ρ	1.3402 ± 0.0028(0.0028)
0.963	C_1^ω	−3.9995 ± 0.0092(0.0106)
	C_2^ω	9.000 ± 0.095(0.109)
	C_1^ϕ	1.9176 ± 0.0061(0.0038)
	C_2^ϕ	3.999 ± 0.056(0.034)
	θ_1^ω[deg]	91.37789 ± 0.00099 (0.00099)
	θ_2^ω[deg]	91.40399 ± 0.00051 (0.00051)
	θ_1^ϕ[deg]	157.99 ± 0.68(0.68)
	θ_2^ϕ[deg]	158.0± 3.7 (3.7)
	λ[GeV]	0.0203 ± 0.0059(0.0060)
	r_E^p[fm]	**0.862 ± 0.002(0.002)**
Polynomial	a_1	−0.2407 ± 0.0047(0.0047)
χ^2/dof	a_2	0.0229 ± 0.0088(0.0089)
0.928	r_E^p[fm]	**0.849 ± 0.008(0.008)**

Table 5.3. Fits to sets of complete data only, weights constracted without correlated normalization errors.

Model name & Fit quality	Free & derived parameters	Estimated parameter values and their errors
Mack-98	x_1^s	$2.03 \pm 0.17(0.15)$
	x_1^v	$1.89 \pm 0.02(0.02)$
	x_2^s	$0.0755 \pm 0.0055(0.0049)$
	x_2^v	$0.0225 \pm 0.0015(0.0013)$
χ^2/dof	C_1^ρ	$-0.2123 \pm 0.0087(0.0086)$
1474.1/780	C_2^ρ	$0.6689 \pm 0.0058(0.0057)$
1.889	C_1^ω	$-1.699 \pm 0.016(0.016)$
	C_2^ω	$-9.37 \pm 0.25(0.25)$
	θ_1^ω[deg]	$91.543 \pm 0.017(0.017)$
	θ_2^ω[deg]	$91.414124 \pm 0.000017(0.000017)$
	λ[GeV]	$0.00105 \pm 0.00040(0.00036)$
	r_E^p[fm]	$\mathbf{0.909 \pm 0.001(0.001)}$
GK-98-2V	η_1^v	$0.00589 \pm 0.00012(0.00007)$
	η_2^v	$0.004263 \pm 0.000023(0.000041)$
	η_1^s	$0.03401 \pm 0.00064(0.00035)$
	η_2^s	$0.01211 \pm 0.00043(0.00040)$
χ^2/dof	C_1^ρ	$-10.47 \pm 0.19(0.10)$
1447.7/780	C_2^ρ	$0.478 \pm 0.021(0.019)$
1.856	C_1^ω	$-9.3037 \pm 0.0041(0.0060)$
	C_2^ω	$1.567 \pm 0.047(0.042)$
	θ_1^ω[deg]	$91.504 \pm 0.011(0.010)$
	θ_2^ω[deg]	$91.415123 \pm 0.000002(0.000003)$
	λ[GeV]	$0.219000 \pm 0.000012(0.000012)$
	r_E^p[fm]	$\mathbf{0.9115 \pm 0.0018(0.0014)}$
GK-98-3V	η_1^v	$0.005887 \pm 0.000028(0.000028)$
	η_2^v	$0.004263 \pm 0.000062(0.000063)$
	η_1^s	$0.03401 \pm 0.00014(0.00014)$
	η_2^s	$0.012101 \pm 0.00044(0.00045)$
χ^2/dof	C_1^ρ	$10.47 \pm 0.21(0.12)$
1447.7/776	C_2^ρ	$0.476 \pm 0.034(0.032)$
1.866	C_1^ω	$-9.31 \pm 0.27(0.16)$
	C_2^ω	$1.571 \pm 0.074(0.072)$
	C_1^ϕ	$0.0015 \pm 0.0823(0.0525)$
	C_2^ϕ	$-0.02 \pm 0.27(0.26)$
	θ_1^ω[deg]	$91.50415 \pm 0.00067\ (0.00067)$
	θ_2^ω[deg]	$91.415123 \pm 0.000001\ (0.000001)$
	θ_1^ϕ[deg]	$157.9 \pm 3.7(3.7)$
	θ_2^ϕ[deg]	$158.0 \pm 17.5\ (17.9)$
	λ[GeV]	$0.2190 \pm 0.0059(0.0060)$
	r_E^p[fm]	$\mathbf{0.911 \pm 0.001(0.001)}$
Polynomial	a_1	$-0.2542 \pm 0.0030(0.0030)$
χ^2/dof	a_2	$0.0372 \pm 0.0050(0.0050)$
2.187	r_E^p[fm]	$\mathbf{0.873 \pm 0.005(0.005)}$

Table 5.4. Fits with set of normalizations determined for each model.

Model name and fit quality	Free and derived parameters	Estimated parameter values and their errors
Mack-98	x_1^s	$1.94 \pm 0.17(0.15) \pm 0.06$(norm)
	x_1^v	$1.85 \pm 0.020(0.019) \pm 0.008$(norm)
	x_2^s	$0.0863 \pm 0.0057(0.0052) \pm 0.0023$(norm)
	x_2^v	$0.0256 \pm 0.0014(0.0013) \pm 0.0006$(norm)
χ^2/dof	C_1^ρ	$-0.2035 \pm 0.0087(0.0086) \pm 0.0026$(norm)
1195.6/780	C_2^ρ	$0.6660 \pm 0.0058(0.0058) \pm 0.0017$(norm)
1.533	C_1^ω	$-1.744 \pm 0.015(0.015) \pm 0.008$(norm)
	C_2^ω	$-8.74 \pm 0.25(0.25) \pm 0.09$(norm)
	θ_1^ω[deg]	$91.519 \pm 0.011(0.011) \pm 0.009$(norm)
	θ_2^ω[deg]	$91.414116 \pm 0.000019(0.000019) \pm 0.000009$(norm)
	λ[GeV]	$0.00229 \pm 0.00067(0.00061) \pm 0.00028$(norm)
	r_E^p[fm]	**$0.8950 \pm 0.0014(0.0013) \pm 0.0009$ (norm)**
Polynomial	a_1	$-0.2582 \pm 0.0030(0.0030) \pm 0.0560$(norm)
χ^2/dof	a_2	$0.0505 \pm 0.0050(0.0050) \pm 0.0695$(norm)
1.603	r_E^p[fm]	**$0.8794 \pm 0.0051(0.0051) \pm 0.0954$ (norm)**
GK-98-2V	η_1^v	$0.005646 \pm 0.000028(0.000029) \pm 0.000280$(norm)
	η_2^v	$0.004363 \pm 0.000068(0.000072) \pm 0.000185$(norm)
	η_1^s	$0.03277 \pm 0.00013(0.00014) \pm 0.00150$(norm)
	η_2^s	$0.01262 \pm 0.00050(0.00054) \pm 0.00155$(norm)
χ^2/dof	C_1^ρ	$9.457 \pm 0.074(0.080) \pm 0.578$(norm)
1177.9/780	C_2^ρ	$0.4516 \pm 0.0075(0.0080) \pm 0.0284$(norm)
1.510	C_1^ω	$-8.213 \pm 0.079(0.084) \pm 0.627$(norm)
	C_2^ω	$1.617 \pm 0.019(0.020) \pm 0.074$(norm)
	θ_1^ω[deg]	$91.50423 \pm 0.00079(0.00079) \pm 0.00862$(norm)
	θ_2^ω[deg]	$91.415123 \pm 0.000001(0.000001) \pm 0.000003$(norm)
	λ[GeV]	$0.219637 \pm 0.000033(0.000033) \pm 0.000500$(norm)
	r_E^p[fm]	**$0.8986 \pm 0.0012(0.0016) \pm 0.0041$ (norm)**
Polynomial	a_1	$-0.2598 \pm 0.0030(0.0030) \pm 0.0560$(norm)
χ^2/dof	a_2	$0.0531 \pm 0.0050(0.0050) \pm 0.0695$(norm)
1.605	r_E^p[fm]	**$0.8822 \pm 0.0051(0.0051) \pm 0.0951$ (norm)**
GK-98-3V	η_1^v	$0.005343 \pm 0.000026(0.000026) \pm 0.000021$(norm)
	η_2^v	$0.003972 \pm 0.000044(0.000044) \pm 0.000020$(norm)
	η_1^s	$0.03281 \pm 0.00013(0.00013) \pm 0.00010$(norm)
	η_2^s	$0.01221 \pm 0.00030(0.00030) \pm 0.00014$(norm)
χ^2/dof	C_1^ρ	$9.285 \pm 0.070(0.069) \pm 0.045$(norm)
1176.1/776	C_2^ρ	$0.3355 \pm 0.0070(0.0069) \pm 0.0033$(norm)
1.516	C_1^ω	$-9.573 \pm 0.074(0.073) \pm 0.046$(norm)
	C_2^ω	$1.778 \pm 0.018(0.018) \pm 0.009$(norm)
	C_1^ϕ	$1.3524 \pm 0.0024(0.0024) \pm 0.0019$(norm)
	C_2^ϕ	$-1.255 \pm 0.041(0.041) \pm 0.015$(norm)
	θ_1^ω[deg]	91.50414 ± 0.00066 $(0.00066) \pm 0.00057$(norm)
	θ_2^ω[deg]	91.415123 ± 0.000001 $(0.000001) \pm 0.000001$(norm)
	θ_1^ϕ[deg]	$162.4313 \pm 0.0011(0.0011) \pm 0.0009$(norm)
	θ_2^ϕ[deg]	$162.370 \pm 0.010(0.010) \pm 0.007$(norm)
	λ[GeV]	$0.224153 \pm 0.000056(0.000056) \pm 0.000023$(norm)
	r_E^p[fm]	**$0.9029 \pm 0.0009(0.0009) \pm 0.0007$ (norm)**
Polynomial	a_1	$-0.2607 \pm 0.0030(0.0030) \pm 0.0560$(norm)
χ^2/dof	a_2	$0.0525 \pm 0.0050(0.0050) \pm 0.0695$(norm)
1.650	r_E^p[fm]	**$0.8838 \pm 0.0051(0.0051) \pm 0.0949$ (norm)**

Table 5.5. Fits with common set of averaged normalizations.

Model name & Fit quality	Free & derived parameters	Estimated parameter values and their errors
Mack-98	x_1^s	$2.03 \pm 0.14(0.10) \pm 0.03$(norm)
	x_1^v	$1.866 \pm 0.017(0.014) \pm 0.005$(norm)
	x_2^s	$0.0819 \pm 0.0046(0.0035) \pm 0.0016$(norm)
	x_2^v	$0.0243 \pm 0.0011(0.0008) \pm 0.0003$(norm)
χ^2/dof	C_1^ρ	$-0.2069 \pm 0.0087(0.0086) \pm 0.0026$(norm)
1199.8/780	C_2^ρ	$0.6682 \pm 0.0058(0.0058) \pm 0.0017$(norm)
1.538	C_1^ω	$-1.737 \pm 0.015(0.015) \pm 0.008$(norm)
	C_2^ω	$-8.88 \pm 0.25(0.25) \pm 0.09$(norm)
	θ_1^ω[deg]	$91.520 \pm 0.011(0.011) \pm 0.009$(norm)
	θ_2^ω[deg]	$91.414116 \pm 0.000019(0.000019) \pm 0.000009$(norm)
	λ[GeV]	$0.00174 \pm 0.00046(0.00035) \pm 0.00017$(norm)
	r_E^p[fm]	$\mathbf{0.8975 \pm 0.0012(0.0010) \pm 0.0008}$ (norm)
GK-98-2V	η_1^v	$0.005690 \pm 0.000028(0.000029) \pm 0.000023$(norm)
	η_2^v	$0.004326 \pm 0.000064(0.000066) \pm 0.000028$(norm)
	η_1^s	$0.03299 \pm 0.00013(0.00013) \pm 0.00011$(norm)
	η_2^s	$0.01239 \pm 0.00047(0.00049) \pm 0.00021$(norm)
χ^2/dof	C_1^ρ	$9.53 \pm 0.17(0.15) \pm 0.08$(norm)
1179.5/780	C_2^ρ	$0.4575 \pm 0.0078(0.0078) \pm 0.0034$(norm)
1.512	C_1^ω	$-8.29 \pm 0.18(0.16) \pm 0.09$(norm)
	C_2^ω	$1.601 \pm 0.024(0.023) \pm 0.011$(norm)
	θ_1^ω[deg]	$91.50425 \pm 0.00075(0.00075) \pm 0.00065$(norm)
	θ_2^ω[deg]	$91.415123 \pm 0.000001(0.000001) \pm 0.000001$(norm)
	λ[GeV]	$0.21963 \pm 0.00019(0.00017) \pm 0.00008$(norm)
	r_E^p[fm]	$\mathbf{0.8999 \pm 0.0014(0.0013) \pm 0.0006}$ (norm)
GK-98-3V	η_1^v	$0.005314 \pm 0.000025(0.000025) \pm 0.000020$(norm)
	η_2^v	$0.003992 \pm 0.000046(0.000046) \pm 0.000021$(norm)
	η_1^s	$0.03267 \pm 0.00012(0.00012) \pm 0.00010$(norm)
	η_2^s	$0.01236 \pm 0.00032(0.00032) \pm 0.00015$(norm)
χ^2/dof	C_1^ρ	$9.27 \pm 0.13(0.25) \pm 0.16$(norm)
1173.8/776	C_2^ρ	$0.334 \pm 0.012(0.023) \pm 0.013$(norm)
1.513	C_1^ω	$-9.56 \pm 0.14(0.27) \pm 0.17$(norm)
	C_2^ω	$1.783 \pm 0.029(0.051) \pm 0.031$(norm)
	C_1^ϕ	$1.3524 \pm 0.0024(0.0024) \pm 0.0018$(norm)
	C_2^ϕ	$-1.261 \pm 0.042(0.042) \pm 0.015$(norm)
	θ_1^ω[deg]	$91.50412 \pm 0.00066\ (0.00066) \pm 0.00057$(norm)
	θ_2^ω[deg]	$91.415123 \pm 0.000001\ (0.000001) \pm 0.000001$(norm)
	θ_1^ϕ[deg]	$162.4313 \pm 0.0011(0.0011) \pm 0.0009$(norm)
	θ_2^ϕ[deg]	$162.371 \pm 0.010(0.010) \pm 0.007$(norm)
	λ[GeV]	$0.224163 \pm 0.000056(0.000056) \pm 0.000023$(norm)
	r_E^p[fm]	$\mathbf{0.9022 \pm 0.0016(0.0030) \pm 0.0021}$ (norm)
Polynomial	a_1	$-0.2597 \pm 0.0030(0.0030) \pm 0.0560$(norm)
χ^2/dof	a_2	$0.0517 \pm 0.0050(\ 0.0050) \pm 0.0695$(norm)
1.630	r_E^p[fm]	$\mathbf{0.8820 \pm 0.0051(0.0051) \pm 0.0951}$ (norm)

We treat this fact as a signal that our estimators are slightly biased because all the fitted models are analytic with a true domain of analyticity and it has been expected that for all reliable fits the interval estimators of the model radii would belong to (or at least, well overlapping) the "polynomial radius" interval estimator.

Now, to show that possible biases are indeed presented in the above estimates, we repeat all the fits on the complete data sets only, but with the weights constructed without admixture of the normalization errors as described below. The results are shown in Table 5.3. They exhibit significant shifts in the adjustable parameters and radii values, however, one should hardly trust these estimates because of a bad quality of fits. The bad fits quality is a signal that either data from different experiments are inconsistent with each other in normalizations or the models under consideration have the limited areas of applicability. So, we will try to construct a self-consistent data sample guided by the models and normalization errors in the data. Then, we will try to find areas of application for each model.

With each of the complete experimental data sets we associate the adjustable renormalization parameter $1/\lambda^i$ to simulate a possible systematic rescaling of the data of different experiments with respect to each other. These normalization parameters are a subject of evaluation by minimizing a least squares χ^2 functional + "penalty function"

$$\chi 2 = \sum_{i=1}^{N} (\vec{y}^{\,i} - \lambda^i \vec{t}^{\,i}) \cdot W^i \cdot (\vec{y}^{\,i} - \lambda^i \vec{t}^{\,i}) + \sum_{i=1}^{N} \sum_{j=1}^{n^i} \frac{(\lambda^i - 1)^2}{\sigma_i(j)^2}, \tag{2}$$

where i is the number of experiment, λ^i is the normalization for i'th experiment, $\vec{y}^{\,i}$ is the vector contained the experimental differential cross section values of the i'th experiment, $\vec{t}^{\,i}$ is the vector contained the calculated cross-section values, W^i is the inverted error matrix for the data of the i'th experiment, n^i is the number of points at the i'th set.

λ_i was calculated analytically at each step of minimization. For details see Appendix 1.

The second term in (2) is the "penalty function" increasing the χ^2 value when the normalizations come out from unity.

Let us denote the total point-to-point normalization error quoted by experimentalists as $\nu_i(j)$, where j is the point number in the i'th data set. Analogously, let $s_i(j)$ be the total systematic error from all other sources combined in quadratures.

We apply the following procedure:

- If systematic errors due to normalization were declared by authors, we use $\sigma_i(j) = \nu_i(j)$;
- If $\nu_i(j)$ is not given, but $s_i(j)$ is given, then $\sigma_i(j) = s_i(j)$, with motivation that in this case we have a reference number and we adopt that $\nu_i(j) \leq s_i(j)$. In such a way we will obtain the upper limit of the normalization systematic errors, since we cannot separate the non-normalization part of error.
- As has been indicated earlier we do not analyze the data which contained only total errors without any information about their sources. Certainly, we lose part of statistics (about 7 %), however, we will obtain more reliable results with full control of parameter errors.

So, at the end of the minimization we would have three sets of normalization factors, one for each model, and the same number of normalization correlation matrices.

The results of the fits with renormalizations are presented in Table 5.4. We obtain the slightly better fits for each model but they all are still bad. We attribute this badness of fits either to the limited areas of the applicability of the models used or to the overestimation of the experimental errors in some experiments.

Let us stress that now we have three different estimates of the radius resulting from the polynomial fits, one for each set of the renormalization parameters. Now, we have the situation when the interval estimator of the radii derived from polynomial fits for each set of normalizations contains the corresponding interval estimator for the model radii (as it should be when there are no rapid oscillations near $t = 0$).

As all the models give a rough equally "good" description, we cannot choose the "best" model and the corresponding set of normalizations among others. To proceed to the determination of the areas of applicability of the models selected we construct a common set of normalizations following the procedure described in the Appendix 1 ("weighted averaging of normalizations", see Table 5.1). We will use this set of normalizations in the subsequent iterations of the fits. Their error matrix[3] will be used in the calculations of the parameter errors induced by our renormalization procedure.

To check that we did not destroy drastically the previous results with three different sets of normalizations, we repeat all the fits with obtained common renormalization parameters.

The results are presented in Table 5.5. As was expected we did get the "slightly worse fits" than those previously obtained with the individual sets of normalizations, but the whole picture of the results remains unchanged. All the three models give practically the same goodness of fit and closer to each other values of the proton electric radii.

6 An attempt to get the confidential areas of applicability

From previous sections we see that the "naive" (all errors quadratically combined) fits to the sample of complete data sets give pretty good chisquares, however, the fits to the same data sample with weights constructed from total errors without admixtures of correlated normalization errors result in unacceptably large chisquares. Using the renormalizations of the data admissable by the correlated normalization errors in the data we slightly improve the quality of the fits but they still remain unacceptable. This may be due to the following reasons or their combination:

- each model has own and restricted areas of applicability which should be determined;
- experimental data have hidden systematics not quoted by experimentalists.

One possibility to get reliable parameter estimates is to use PDG scale factor prescription [12] and to inflate parameter errors with scale-factor resulted from the fits (but not the errors due to the play with normalizations). But this way says nothing about the area of models applicability and may be relevant only if the areas of applicability are known.

So, we have to determine the areas of applicability first or discover the subsample of data which has the overestimated experimental errors. The way that can help to identify the areas of applicability of each model is the following "model driven filtration procedure."[19]

- The whole physical region of kinematic variables $\log q^2$ versus $\cos\theta$[4] is divided into bins and each parametrization is fitted to the data set using normalizations obtained above.
- The value of χ^2/N_{points} is evaluated for each non-empty bin, where N_{points} is the number of points per bin. The bin is declared as belonging to "the good description area" (GDA) for a particular model if this value is less than some threshold value. Otherwise, this bin is declared as belonging to "the bad description area" (BDA). The threshold normalized

[3] Final arrays of normalizations and their error matrices obtained for each model and their "weighted averages" can be obtained by contacting polishchuk@mx.ihep.su.

[4] In these variables the distribution of points is sufficiently uniform.

χ^2 value is chosen in such a way that the confidence level (CL) calculated for "good description" subset would exceed some threshold value (say 50% or 90%). The procedure starts from sufficiently large $\kappa = \chi^2/N_{bin} = 5$, so at the first iteration all the points belong the GDA. Then, we decrease κ by small steps, calculate CL at each value of the κ and determine the corresponding temporal GDA subsets until the threshold value is reached.

- The "good" and "bad" data subsets are formed from the bins classified in such a way. The "good" subset includes the points belonging to GDA of all models and, similarly, the "bad" subset contains the points belonging to BDA of all models.

The results of our standard fits of all models on the "good at 90% CL" data sample are presented in Table 6.1. We see that rejection the "bad" data points not much affected on the values of the parameters for all models. The interval estimators are slightly biased but the joint interval estimate for the radius before filtration and after filtration are well overlapped.

Before the model driven filtration, we have

$$r_E^p \in [0.8912, 0.9021] \quad [\text{fm}] \quad \text{three models},$$

$$r_E^p \in [0.7818, 0.9822] \quad [\text{fm}] \quad \text{polinomial}.$$

After the model driven filtration, we get

$$r_E^p \in [0.8955, 0.9059] \quad [\text{fm}] \quad \text{three models},$$

$$r_E^p \in [0.7847, 0.9885] \quad [\text{fm}] \quad \text{polinomial}.$$

From these interval estimators (constructed using a linear combination of the experimental and normalization errors), we see that the polynomial estimator is inconclusive because it gives too wide interval estimator.

To show the stability of the radii under variation of the threshold CL value used in classifying data points as "good" and "bad", the radii obtained at two values of the CL = 50 % and CL=90 % are tabulated below.

Model	50% CL, 713 points r_E^p in [fm]	90% CL, 696 points r_E^p in [fm]
Mack-98	0.8963 ± 0.0017	0.8937 ± 0.0015
GK-98-2V	0.8976 ± 0.0011	0.8971 ± 0.0016
GK-98-3V	0.9008 ± 0.0011	0.9005 ± 0.0010
Polynomial	0.8842 ± 0.0058	0.8866 ± 0.0062

The quoted errors do not include the contributions due to normalization errors.

Now, let us discuss further the areas of applicability. If the subset of bad data is not empty and is grouped in such a way that this subset can be interpreted as the area where all models failed then the good data subset belongs to the area of applicability with a preselected CL.

If the bad data points are randomly distributed over the whole available kinematic region, then we have only one possibility: To declare that the models are applicable in the whole region and the bad data are the data with overestimated errors (occasionally outstanding data). In this case, in order to estimate the errors in the parameters, we should use the Birge scale-factor method. The situation with bad data bins is presented in Fig.3. There are no prominent regularities in the population of bad data bins in the whole kinematic region and we conclude that our models are applicable in the whole kinematic region, except for a few bins with presumably overestimated experimental errors. The only way to get reliable errors is the usage of the scale factor prescription. As the best estimate of the proton electric radius in terms of average, dispersion and systematic errors, we take the estimate derived from the GK-98-2V fit, taking

into account the scale factor $\sqrt{1.512}$ (see Table 5.5), and assign the additional systematic error due to model selection as one half of the maximal difference between central values of the radii. So, we have

$$r_E^p = 0.900 \pm 0.002(exp) \pm 0.001(norm) \pm 0.002(models) \quad [\mathrm{fm}].$$

The same procedure of constructing a final radius estimate based on the filtered data (see Table 6.1) gives

$$r_E^p = 0.897 \pm 0.002(exp) \pm 0.001(norm) \pm 0.003(models) \quad [\mathrm{fm}].$$

We see that the estimate obtained with the model driven filtration gives the result consistent with that obtained by using the Birge factor. So, our best estimate of the proton radius is the estimate based on the model driven filtration.

Model name & Fit quality	Free & derived parameters	Estimated parameter values and their errors
Mack-98	x_1^s	$1.59 \pm 0.17(0.13) \pm 0.039$(norm)
	x_1^v	$1.806 \pm 0.020(0.017) \pm 0.0056$(norm)
	x_2^s	$0.0953 \pm 0.0057(0.0045) \pm 0.0018$(norm)
	x_2^v	$0.0284 \pm 0.0014(0.0010) \pm 0.0004$(norm)
χ^2/dof	C_1^ρ	$-0.213 \pm 0.011(0.011) \pm 0.003$(norm)
510.3/685	C_2^ρ	$0.6722 \pm 0.0078(0.0077) \pm 0.0019$(norm)
0.745	C_1^ω	$-1.725 \pm 0.020(0.020) \pm 0.007$(norm)
	C_2^ω	$-9.07 \pm 0.30(0.30) \pm 0.09$ (norm)
	θ_1^ω[deg]	$91.521 \pm 0.014(0.014) \pm 0.009$(norm)
	θ_2^ω[deg]	$91.414116 \pm 0.000021(0.000021) \pm 0.000008$(norm)
	λ[GeV]	$0.0035 \pm 0.0009(0.0007) \pm 0.0003$(norm)
	r_E^p[fm]	$\mathbf{0.8937 \pm 0.0015(0.0013) \pm 0.0008}$ **(norm)**
GK-98-2V	η_1^v	$0.005652 \pm 0.000032(0.000032) \pm 0.000021$(norm)
	η_2^v	$0.004407 \pm 0.000077(0.000077) \pm 0.000027$(norm)
	η_1^s	$0.03275 \pm 0.00015(0.00015) \pm 0.00011$(norm)
	η_2^s	$0.01253 \pm 0.00055(0.00056) \pm 0.00019$(norm)
χ^2/dof	C_1^ρ	$9.25 \pm 0.19(0.15) \pm 0.08$(norm)
493.9/685	C_2^ρ	$0.457 \pm 0.010(0.010) \pm 0.003$(norm)
0.721	C_1^ω	$-7.98 \pm 0.20(0.17) \pm 0.08$(norm)
	C_2^ω	$1.594 \pm 0.030(0.030) \pm 0.011$(norm)
	θ_1^ω[deg]	$91.50427 \pm 0.00094(0.00095) \pm 0.00068$(norm)
	θ_2^ω[deg]	$91.415123 \pm 0.000001(0.000001) \pm 0.000001$(norm)
	λ[GeV]	$0.21991 \pm 0.00023(0.00019) \pm 0.00008$(norm)
	r_E^p[fm]	$\mathbf{0.8971 \pm 0.0016(0.0014) \pm 0.0008}$ **(norm)**
GK-98-3V	η_1^v	$0.005844 \pm 0.000032(0.000030) \pm 0.000021$(norm)
	η_2^v	$0.004495 \pm 0.000073(0.000072) \pm 0.000024$(norm)
	η_1^s	$0.03358 \pm 0.00015(0.00014) \pm 0.00010$(norm)
	η_2^s	$0.01253 \pm 0.00050(0.00050) \pm 0.00016$(norm)
χ^2/dof	C_1^ρ	$9.877 \pm 0.077(0.072) \pm 0.036$(norm)
494.8/681	C_2^ρ	$0.4161 \pm 0.0089(0.0084) \pm 0.0028$(norm)
0.726	C_1^ω	$-8.798 \pm 0.082(0.076) \pm 0.036$(norm)
	C_2^ω	$1.688 \pm 0.022(0.021) \pm 0.009$(norm)
	C_1^ϕ	$0.1937 \pm 0.0053(0.0053) \pm 0.0034$(norm)
	C_2^ϕ	$-0.137 \pm 0.072(0.072) \pm 0.020$(norm)
	θ_1^ω[deg]	$91.50409 \pm 0.00085(0.00085) \pm 0.00062$(norm)
	θ_2^ω[deg]	$91.415123 \pm 0.000001\ (0.000001) \pm 0.000001$(norm)
	θ_1^ϕ[deg]	$157.8 \pm 2.6(2.6) \pm 1.9$(norm)
	θ_2^ϕ[deg]	$159.9 \pm 13.0(13.0) \pm 8.0$(norm)
	λ[GeV]	$0.219171 \pm 0.000084(0.000085) \pm 0.000038$(norm)
	r_E^p[fm]	$\mathbf{0.9005 \pm 0.0010(0.0009) \pm 0.0006}$ **(norm)**
Polynomial	a_1	$-0.2624 \pm 0.0036(0.0036) \pm 0.0567$(norm)
χ^2/dof	a_2	$0.0575 \pm 0.0059(0.0059) \pm 0.0697$(norm)
0.817	r_E^p[fm]	$\mathbf{0.8866 \pm 0.0062(0.0062) \pm 0.0957}(\boldsymbol{norm})$

Table 6.1. Results of the last iteration of fits on filtered data sample.

7 Summary

We have made the comparison of the majority of published parametrizations of the proton electromagnetic FFs using the largest experimental data sample on $d\sigma/dt$ ever used. All the models give unacceptable "naive" fit quality to the whole data sample and we have to select and modify three parametrizations for further detailed analyses. These three hybrid parametrizations give the best and close to each other values of the $\chi 2/d.o.f.$ parameter which characterizes the

fit quality (for the "naive" fits it is less than 1.2 for each model, see Table 4.1).

All the reconstructed models have correct analytical structures with the minimal set of known singularities. They are all easily generalized for inclusion of the additional Wataghin terms [15],[22] into the GVMD multiplier of the FF models for additional vector mesons as is seen from inspection formulas for the GK-98-2V and GK-98-3V parametrizations. It should be noted that the addition of ϕ-meson did not resulted in improving the fits quality to the data in the space-like region (remember, that the addition of ϕ-meson gives four extra adjustable parameters), but shifted slightly the proton radius towards higher values.

The common feature of the tree models used is the presence of pionium contribution. In our preparatory attempts to produce cross assessments with models without these contributions, we had to reject by all the filtrations more experimental data than in selected and modified models with pionium. In the case with pionium all the filtrations rejected less than 18 %. It is strong evidence in favour of the pionium contribution to the GVDM multiplier of the FFs. The exact form of pionium contributions to the FFs is the task for the future refinements of our procedure and models.

It should be stressed that in the recent experiment conducted by L.L.Nemenov with coworkers at IHEP (Protvino) the pionium was detected experimentally [29].

As it is seen from Tables 5.5 and 6.1 all three models give statistically equivalent descriptions of the unfiltered and filtered data. It can be also seen in Figs. 4 - 5, where the modulae of the electric and magnetic formfactors for each model are presented for comparison in a large interval of negative and positive q^2.

The model GK-98-2V is preferable at this stage of data and models cross assessment because it has a proper behaviour at high q^2 and gives the best description of the assessed data sample. The main results are summarized as follows:

- We have developed *the data and models cross assessments (CRAS) technology* in the case of the data on elastic $e^{\pm}p$ scattering and models of the nucleon form factors. We believe that the CRAS technology will be applicable in other areas as well and will be helpful in the following tasks:
  - to make data from different experiments mutually consistent on the basis of available relevant models;
  - to determine the areas of applicability of the competitive models;
  - to produce technical and derived physical parameters estimations with metrological quality.
- Using the CRAS technology we have constructed, from the well known inputs, the currently best parametrization of the nucleon FFs in the space-like region which is extendable in the sense of GVDM, can be also used to describe nucleon FFs in the time-like region, and is capable to eliminate the "false" contradiction between the charge radius of the proton from the hydrogen Lamb shift measurements (see recent review of S.G. Karshenboim [30]) and from the high-energy electron-proton elastic scattering data.

From the fit results of all the models to the "90% CL" sample, we have derived our best estimate of the proton charge radius

$$r_E^p = 0.897 \pm 0.002(exp) \pm 0.001(norm) \pm 0.003(models) \quad [\mathrm{fm}]$$

given by the modified Gary-Krümpelman model on the filtered data sample. This estimate is in good agreement with the estimate derived from the fits to unfiltered data but with the usage of the Birge method to justify the errors assigned to the adjustable parameters

$$r_E^p = 0.900 \pm 0.002(exp) \pm 0.001(norm) \pm 0.002(models) \quad [\mathrm{fm}].$$

The traditional method of extracting proton radius from quadratic polynomial fit gives the inconclusive estimates on the model driven renormalized data (see Table 5.5 and Table 6.1), because they have too large systematic errors due to more correct treatment of the systematic error in the radius induced by the correlated experimental errors of normalization.

It should be noted that the above results on the parameters of the three models analysed are still preliminary. They are a subject for the re-estimation with the data on $n\ e^-$, $e^{\pm}\ n$ elastic scattering and $e^+e^- \to$ *nucleon anti-nucleon* added to the data on $e^{\pm}\ p$ elastic scattering. It is likely that all models will require the inclusion of the protonium atom singularities. We plan to perform further CRAS-iterations for the nucleon form factors with an extended data sample in the near future.

A straightforward outcome of our analyses for the current and planned experiments are as follows:

- The HERA data on $e^{\pm}p \to e^{\pm}p$ would be extremely useful not only for the extraction of the FFs but also to check the relevancy of one photon approximation in the description of the $d\sigma/dt$ data at moderate $-t$. Unfortunately, these measurements are not discussed or planned in the HERA physics program.

- The CEBAF data near $t = 0$ will be very helpful to test the GVDM with pionium contribution (note that "bad" data points are concentrated near $t = 0$, see Fig.3.)

To simplify the reproduction of our results, we prepare a special "WEB $e^{\pm}p \to e^{\pm}p$ data site" with computer readable files of all data, models, and current results for the arrays of parameter values and their correlators on the basis of PPDS data. It can be reached by

\protect\vrule width0pt\protect\href{http://sirius.ihep.su\string~polishch/ep

or

\protect\vrule width0pt\protect\href{http://dbserv.ihep.su\string~bvp/ep2ep_t

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A Normalizations and their correlations

We shall use the following notation:

$\vec{y}^{\ i}$ — array of average values of an observable $y(x)$ measured at n^i x-points in i-th experiment.

T^i — $n^i \times n^i$ error matrix of i-th experiment.

$W^i = (T^i)^{-1}$ — weight matrix for the data of i-th experiment.

$\nu_i(j)$ — estimation of normalization uncertainty in j-th point of the i-th experiment.

λ^i — auxiliary normalization factor to be assigned to i-th experiment to tune mutual normalizations of different independent experiments.

We shall use the following expression for $\chi 2$

$$\chi 2 = \sum_{i=1}^{N} (\vec{y}^{\ i} - \lambda^i \vec{t}^{\ i}) \cdot W^i \cdot (\vec{y}^{\ i} - \lambda^i \vec{t}^{\ i}) + \sum_{i=1}^{N} \sum_{j=1}^{n^i} \frac{(\lambda^i - 1)^2}{\nu_i(j)^2}, \tag{1}$$

which must be minimized with respect to model parameters $\vec{p}^{\ m}$ and with respect to auxiliary normalization factors. Necessary requirements for extremums lead to equations

$$\frac{\partial \chi 2^m}{\partial p^m_{\alpha_m}} = -2 \sum_{i=1}^{N} \lambda^{m,i} \frac{\partial \vec{t}^{\ m}(i)}{\partial p^m_{\alpha_m}} \cdot W^i (\vec{y}^{\ i} - \lambda^{m,i} \vec{t}^{\ m}(i)) = 0, \tag{2}$$

$$\frac{\partial \chi 2^m}{\partial \lambda^{m,i}} = -2 \vec{t}^{\ m}(i) \cdot W^i (\vec{y}^{\ i} - \lambda^{m,i} \vec{t}^{\ m}(i)) + 2 \sum_{j=1}^{n^i} \frac{\lambda^{m,i} - 1}{\nu_i(j)^2} = 0. \tag{3}$$

From the second set of equations one can express auxiliary normalizations for m-th model as follows:

$$\lambda^{m,i} = \frac{\vec{t}^{\ m}(i) \cdot W^i \vec{y}^{\ i} + \sum_{j=1}^{n^i} 1/\nu_i(j)^2}{\vec{t}^{\ m}(i) \cdot W^i \vec{t}^{\ m}(i) + \sum_{j=1}^{n^i} 1/\nu_i(j)^2} \ . \tag{4}$$

Inserting these expressions for normalization factors into above expression for $\chi 2^m$, we will get a new formula for $\chi 2^m$, which must be minimized with respect to model parameters only. After solving the minimization task and selecting proper local minimum point $\langle p^m_{\alpha_m} \rangle$, we can obtain the corresponding error matrix

$$\| \frac{1}{2} \frac{\partial \chi 2^m}{\partial p^m_{\alpha_m} \partial p^m_{\beta_m}} \|^{-1} \ = \langle \Delta p^m_{\alpha_m} \Delta p^m_{\beta_m} \rangle \ = \ \Pi^m_{\alpha_m \beta_m}.$$

In a general case the normalization parameters for different models are correlated, and some of the normalizations may be strongly correlated for one model. Let us first prepare formulae for correlators between normalizations for different models.

$$N^{mm'}_{ij} = \langle (\sum_{\alpha_m} \frac{\partial \lambda^{m,i}}{\partial p^m_{\alpha_m}} \Delta p^m_{\alpha_m} + \sum_{\mu_i} \frac{\partial \lambda^{m,i}}{\partial y^i_{\mu_i}} \Delta y^i_{\mu_i})(\sum_{\beta_{m'}} \frac{\partial \lambda^{m',j}}{\partial p^{m'}_{\beta_{m'}}} \Delta p^{m'}_{\beta_{m'}} + \sum_{\mu_j} \frac{\partial \lambda^{m',j}}{\partial y^j_{\mu_j}} \Delta y^j_{\mu_j}) \rangle$$

$$= \sum_{\alpha_m \beta_{m'}} \frac{\partial \lambda^{m,i}}{\partial p^m_{\alpha_m}} \frac{\partial \lambda^{m',j}}{\partial p^{m'}_{\beta_{m'}}} \Pi^{mm'}_{\alpha_m \beta_{m'}} + \sum_{\beta_{m'} \mu_i} \frac{\partial \lambda^{m,i}}{\partial y^i_{\mu_i}} \frac{\partial \lambda^{m',j}}{\partial p^{m'}_{\beta_{m'}}} \frac{\partial p^{m'}_{\beta_{m'}}}{\partial y^i_{\mu_i}} (\sigma^i_{\mu_i})^2$$

$$+ \sum_{\alpha_m \mu_j} \frac{\partial \lambda^{m,i}}{\partial p^m_{\alpha_m}} \frac{\partial \lambda^{m',j}}{\partial y^j_{\mu_j}} \frac{\partial p^m_{\alpha_m}}{\partial y^j_{\mu_j}} (\sigma^j_{\mu_j})^2 + \delta_{ij} \cdot \sum_{\mu_i} \frac{\partial \lambda^{m,i}}{\partial y^i_{\mu_i}} \frac{\partial \lambda^{m',i}}{\partial y^i_{\mu_i}} (\sigma^i_{\mu_i})^2, \tag{8}$$

where

$$\Pi^{mm'}_{\alpha_m \beta_{m'}} = \frac{1}{4} \sum_{\eta,\nu} \Pi^m_{\alpha_m \eta} \Pi^{m'}_{\beta_{m'} \nu} \sum_{i,\mu_i} \frac{\partial^2 \chi 2^m}{\partial p^m_\eta \partial y^i_{\mu_i}} \frac{\partial^2 \chi 2^{m'}}{\partial p^{m'}_\nu \partial y^i_{\mu_i}} (\sigma^i_{\mu_i})^2,$$

and partial derivatives are estimated at minima points for m and m' models. Now we are able to construct estimators for a unique set of lambdas and their error matrix

$$\langle \lambda^i \rangle = (\sum_m \frac{1}{N^{mm}_{ii}})^{-1} \sum_m \frac{\lambda^{m,i}}{N^{mm}_{ii}} \tag{9}$$

and

$$N_{ij} = (\sum_m \frac{1}{N^{mm}_{ii}})^{-1} (\sum_m \frac{1}{N^{mm}_{jj}})^{-1} \sum_{mm'} \frac{N^{mm'}_{ij}}{N^{mm}_{ii} N^{m'm'}_{jj}} \ . \tag{10}$$

B Error propagation

B.1 Errors of model parameters

Having numerical values for normalization correction factors, we can find a final parameter estimation. We will use the standard expression for modified $\chi 2^m$

$$\chi 2^m = \sum_{i=1}^{N} (\vec{y}^{\ i} - \lambda^{m,i} \vec{t}^{\ m}(i)) \cdot W^i (\vec{y}^{\ i} - \lambda^{m,i} \vec{t}^{\ m}(i)),$$

with axillary correction factors fixed at their best values obtained above. After minimization we will obtain new estimations for model parameters and corresponding parameter errors

$$\langle p^m_{\alpha_m} \rangle, \ \langle \Delta p^m_{\alpha_m} \Delta p^m_{\beta_m} \rangle^{expt} = \Pi^m_{\alpha_m \beta_m}.$$

To estimate the contribution to parameters covariance from renormalization, we will propagate covariances N^m_{ij} as follows:

$$\Pi^{m,norm}_{\alpha_m \beta_m} = \langle \Delta p^m_{\alpha_m} \Delta p^m_{\beta_m} \rangle^{norm} = \sum_{ij}^{N} \frac{\partial p^m_{\alpha_m}}{\partial \lambda^i} N^m_{ij} \frac{\partial p^m_{\beta_m}}{\partial \lambda^j}.$$

Following the same way, we can propagate systematic (or statistical) point-to-point errors to be the errors of parameters:

$$\Pi^{m,sys}_{\alpha_m \beta_m} = \langle \Delta p^m_{\alpha_m} \Delta p^m_{\beta_m} \rangle^{sys} = \sum \frac{\partial p^m_{\alpha_m}}{\partial y^i_{\mu_i}} N^{\mu_i \nu_j}_{ij} \frac{\partial p^m_{\beta_m}}{\partial y^j_{\nu_j}},$$

here μ_i and ν_j are the μ'th point in the i'th experiment and ν'th point in the j'th experiment, respectively. If usual MINUIT assumptions about the minimum are valid, this matrix must coincide with the MINUIT error matrix. Really, we have at the minimum position

$$\Delta\chi^2 = \frac{1}{2}\frac{\partial\chi 2}{\partial p_\alpha \partial p_\beta}\langle\Delta p_\alpha \Delta p_\beta\rangle^{calc},$$

in the absence of point-to-point correlation

$$\langle\Delta p_\alpha \Delta p_\beta\rangle^{calc} = \sum \frac{\partial p_\alpha}{\partial y^i_{\mu_i}}\frac{\partial p_\beta}{\partial y^i_{\mu_i}}(\sigma_i^{\mu_i})^2,$$

$$\Delta\chi^2/n_{points} = \frac{1}{2}\frac{\partial\chi 2}{\partial p_\alpha \partial p_\beta}\frac{\partial p_\alpha}{\partial y^i_{\mu_i}}\frac{\partial p_\beta}{\partial y^i_{\mu_i}}(\sigma_i^{\mu_i})^2 = \frac{1}{2}\frac{\partial\chi 2}{\partial y^i_{\mu_i}\partial y^i_{\mu_i}} = 1,$$

it corresponds to the standard MINUIT error definition.

To calculate partial derivatives of parameters over lambdas, we will use the usual trick[5] [31]. The set of equations

$$\frac{\partial\chi 2^m}{\partial p^m_{\alpha_m}} = 0$$

might be considered as the set of functional equations defining dependencies of parameters upon lambdas. Taking the total derivative from both parts of above equations over $\lambda^{m,i}$, we will obtain a linear system

$$\sum_{\beta_m}\left(\frac{\partial^2\chi 2^m}{\partial p^m_{\alpha_m}\partial p^m_{\beta_m}}\frac{\partial p^m_{\beta_m}}{\partial\lambda^{m,i}}\right) + \frac{\partial^2\chi 2^m}{\partial p^m_{\alpha_m}\partial\lambda^{m,i}} \;=\; 0,$$

from which

$$\frac{\partial p^m_{\alpha_m}}{\partial\lambda^{m,i}} \;=-\; \sum_\beta \frac{1}{2}\langle\Delta p^m_{\alpha_m}\Delta p^m_{\beta_m}\rangle^{expt}\frac{\partial^2\chi 2^m}{\partial p^m_{\beta_m}\partial\lambda^{m,i}}\;.$$

Finally, we will have

$$\Pi^{m,norm}_{\alpha_m\beta_m} \;=\; \frac{1}{4}\sum_{ij}^{N}\sum_{\mu\nu}\Pi^m_{\alpha_m\mu}\frac{\partial^2\chi 2^m}{\partial\lambda^{m,i}\partial p^m_\mu}N^m_{ij}\frac{\partial^2\chi 2^m}{\partial\lambda^{m,j}\partial p^m_\nu}\Pi^m_{\alpha_m\nu}$$

with

$$\frac{\partial^2\chi 2^m}{\partial\lambda^{m,i}\partial p^m_\mu} \;=\; 2\frac{\partial\vec{t}^{\,m}(i)}{\partial p^m_\mu}\cdot W^i(2\lambda^{m,i}\vec{t}^{\,m}(i) - \vec{y}^{\,i}).$$

Thus, for each model t^m we will have full estimations of parameters and their covariances

$$\langle p^m_\alpha\rangle,\ \Pi^m_{\alpha_m\beta_m},\ \Pi^{m,norm}_{\alpha_m\beta_m}.$$

[5]We thank our colleague S.I. Alekhin who point out this method

B.2 Errors of proton charge radius

Errors of the proton charge radius are calculated by propagation of the parameter errors of each type separately. The "systematic" error due to the uncertainties in normalizations

$$\Delta^2_{m,(norm)} = \sum_{\alpha_m \beta_m} \frac{\partial r_E^{p,m}}{\partial p^m_{\alpha_m}} \Pi^{m,norm}_{\alpha_m \beta_m} \frac{\partial r_E^{p,m}}{\partial p^m_{\beta_m}} .$$

The error due to the "total experimental" parameters errors obtained from MINUIT system

$$\Delta^2_{m} = \sum_{\alpha_m \beta_m} \frac{\partial r_E^{p,m}}{\partial p^m_{\alpha_m}} \Pi^{m}_{\alpha_m \beta_m} \frac{\partial r_E^{p,m}}{\partial p^m_{\beta_m}} .$$

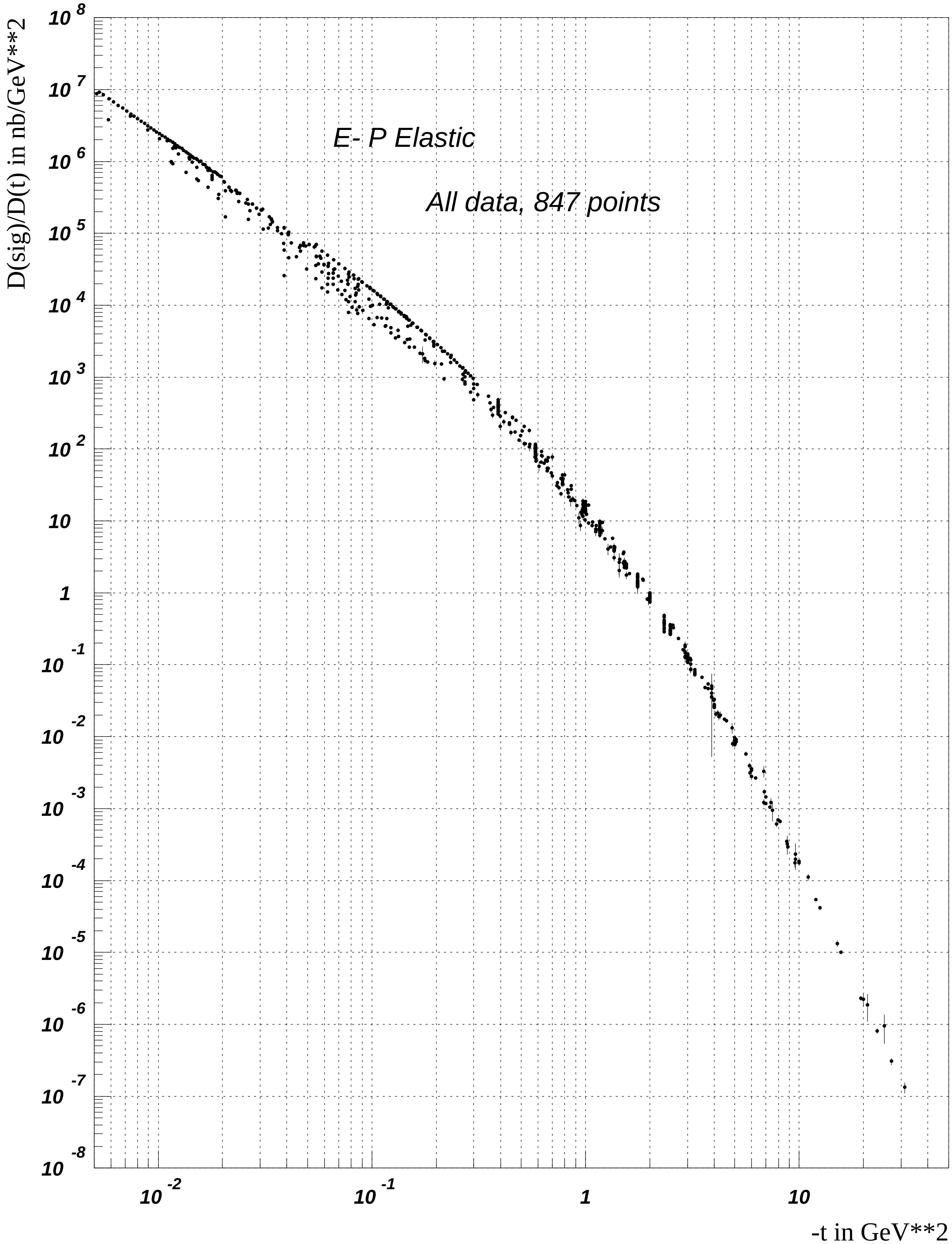

Figure 1: Data atlas.

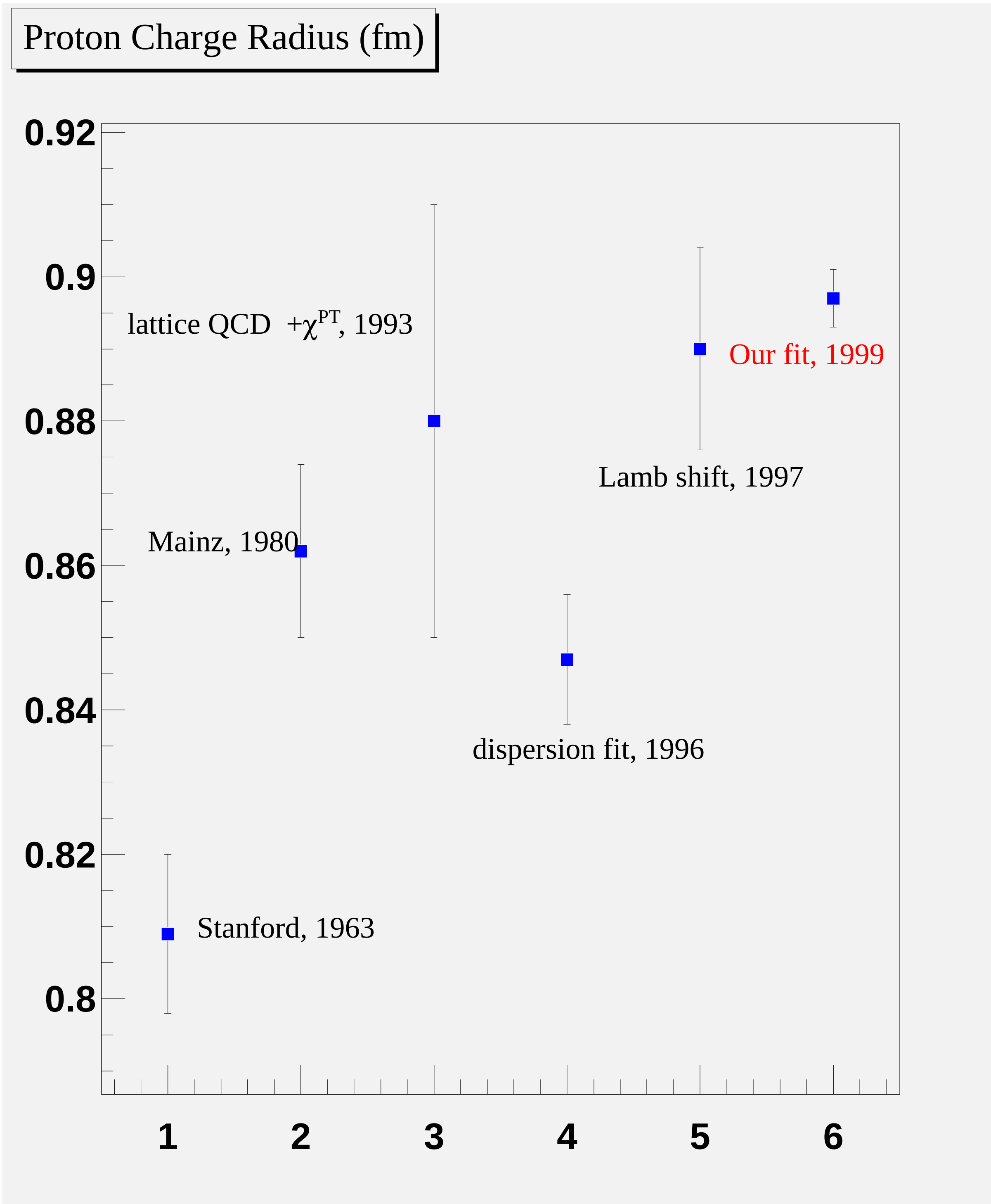

Figure 2: The proton charge radius as it is usually known. (The original picture was in the article of S.S. Karshenboim, see hep-ph/9712347)

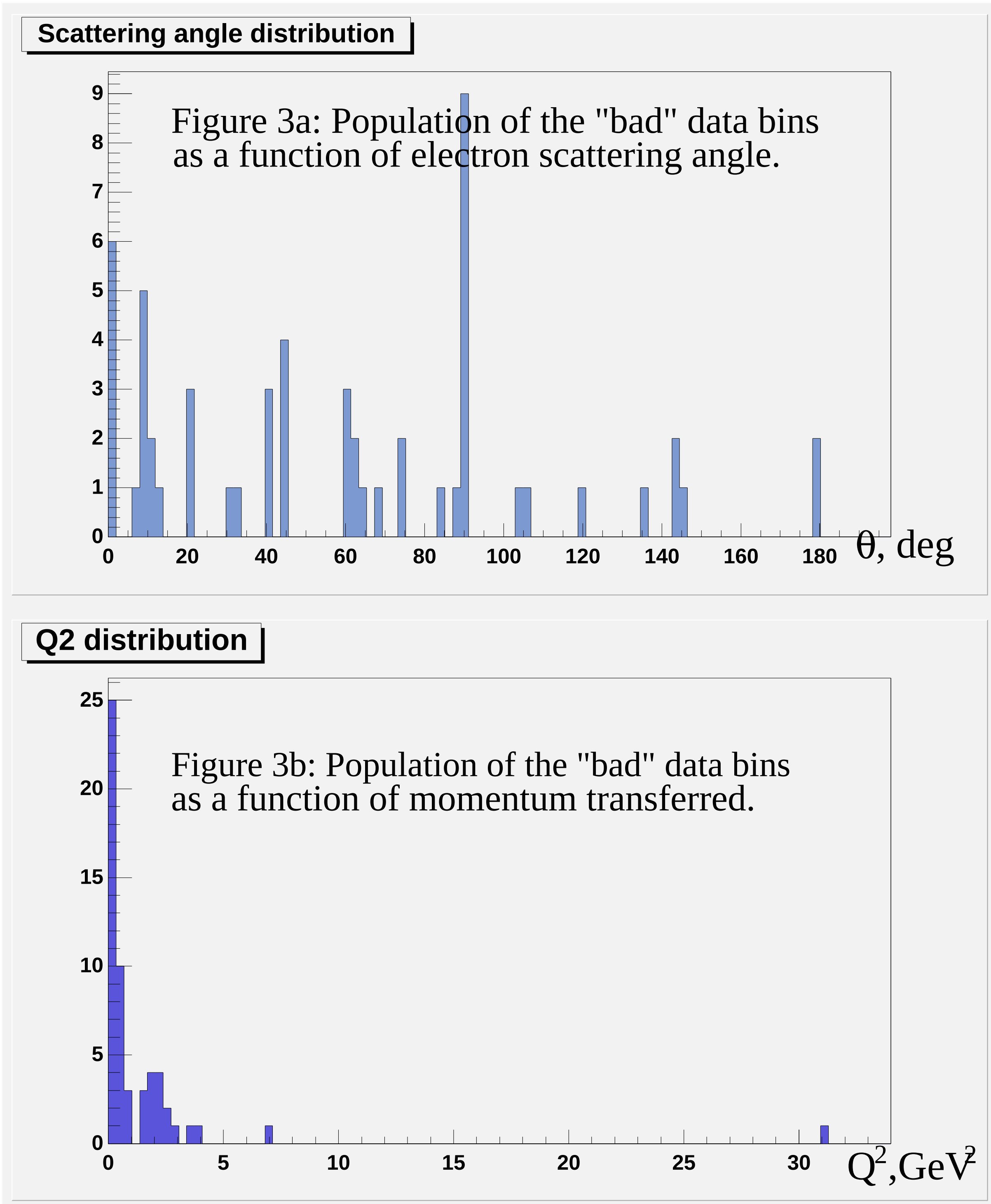

Figure 3: Population of the bad data bins in the whole kinematic region.

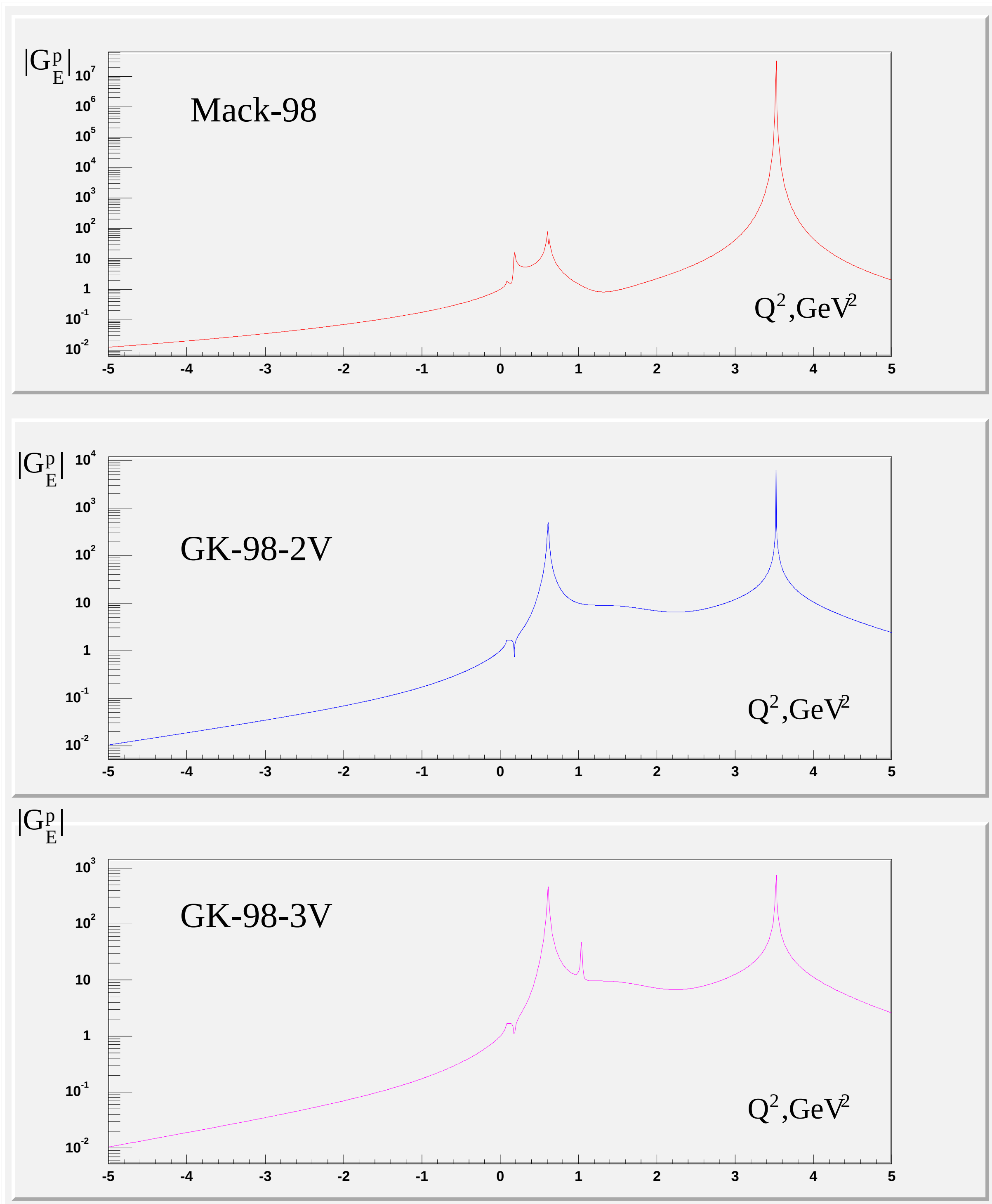

Figure 4: Modulus of the proton charge FFs given by three adjusted models.

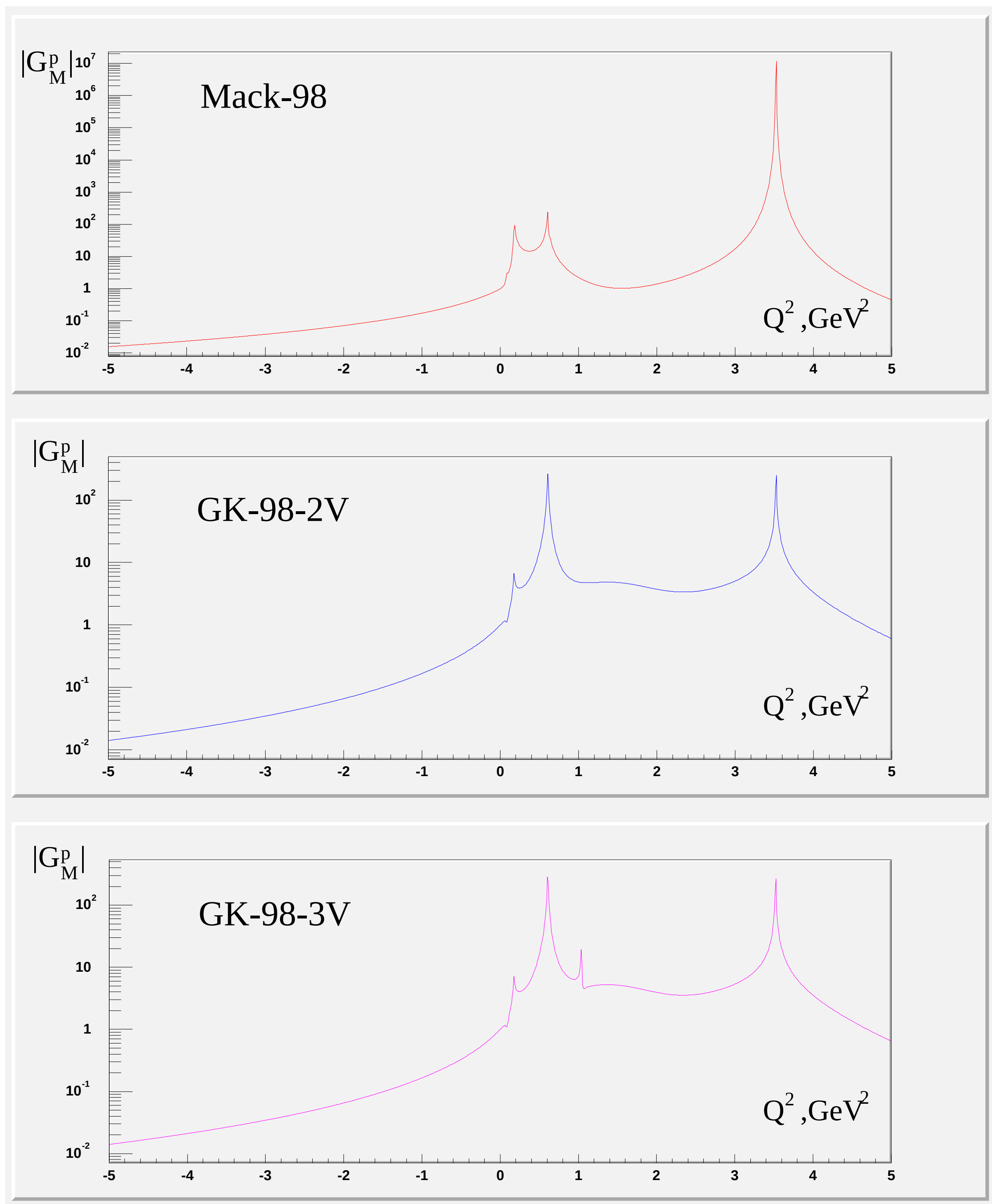

Figure 5: Modulus of the neutron magnetic FFs given by three adjusted models.